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Existence uniqueness and stability for certain operators of nonlinear system of differential equations

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ABSTRACT

This research contributes to the understanding of nonlinear systems of differential equations with operators, specifically in the context of generalizing Volterra and Fredholm integral equations. The use of the Picard approximation method, Banach fixed point theorem, and stability analysis further enhances the analysis of the solutions. The examples provided help to solidify the theoretical findings and highlight their applicability. The outcomes illustrate the efficacy of these tools in envisaging and modelling complex social phenomena. To tackle multifaceted societal challenges, future research in this field should prioritize interdisciplinary collaborations. It is crucial to incorporate empirical data into nonlinear models to validate theoretical findings and enhance practical relevance.

1. Introduction

The study of nonlinear systems of integro-differential equations involves understanding the existence and uniqueness of solutions, stability analysis, and approximation methods [1]–[3]. Researchers use various analytical and numerical techniques to tackle these equations and explore their properties. The findings help deepen our understanding of dynamic systems and provide valuable insights into real-world phenomena [4]–[6].

Indeed, Picard's existence, uniqueness, and stability theorems for first and second-order differential equations are of great importance in the field of ordinary differential equations [7]. These theorems provide fundamental results for establishing the existence and uniqueness of solutions to differential equations and can be extended to higher-order differential equations as well as systems of differential equations [8], [9]. The authors mentioned, such as [10]–[12], have created and developed Banach fixed point theorems to investigate solutions of integro-differential equations in various systems. These theorems provide powerful tools for studying the existence and properties of solutions in these types of equations. The Picard approximation method and the Banach fixed point theorem are particularly useful as they make extensive use of topological concepts [13]–[15], providing a convenient means of qualitatively investigating solutions, periodic solutions, and stability solutions [16]. They also lend themselves well to the utilization of digital computers, which can be versatile tools for finding and constructing approximate solutions and periodic solutions .The numerical-analytic method, proposed by Samoilenko [17], has been employed by many researchers [18], [19] to study and approximate periodic solutions for systems of nonlinear differential equations. This method is widely used in investigating the solvability of nonlinear integro-differential equations and constructing approximate solutions for various systems [17], [20].

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Moreover, the development and application of Banach fixed point theorems, along with the utilization of the Picard approximation method and numerical-analytic methods, have facilitated the study of solutions, periodic solutions, and stability solutions in integro-differential equations. These methods offer convenient approaches for both qualitative and quantitative analysis and have proven to be effective in numerous research studies.

2. Method

2.1. Problem Statement and Fundamental tools

Consider the following nonlinear system of differential equation contain operators which has the forms :

$$\left.\begin{aligned} \frac{dx}{dt} &= f\left(t,x,y,Ax,By\right) \\ \frac{dy}{dt} &= g\left(t,x,y,Ax,By\right) \end{aligned}\right\} \tag{1}$$

With $x(0) = x_0\ , y(0) = 0$

Where A,B are operators define as generalization of Volterra and Fredholm integral equations, $f(t,x,y,z,w)$ *and* $g(t,x,y,z,w)$ are continuous vector functions which are defined on the domains.

$$\left(t,x,y,z,w\right) \in R^1 \times D \times D_1 \times D_2 \times D_3 \tag{2}$$

Where $x \in D \subset R^n\ , D, D_1, D_2, D_3$ are closed and bounded domain subset of Euclidean space R^n . Suppose that the vector functions $f(t,x,y,z,w)$, $g(t,x,y,z,w)$ and the operators A and B are generalizing of Volterra and Fredholm integral equation satisfy the following inequalities.

$$\left.\begin{aligned} \|f(t,x,y,z,w)\| &\le M_1 \\ \|g(t,x,y,z,w)\| &\le N_1 \end{aligned}\right\} \tag{3}$$

$$\left\|f\left(t,x_1,y_1,z_1,w_1\right) - f\left(t,x_2,y_2,z_2,w_2\right)\right\| \le K_1\|x_1 - x_2\| + K_2\left\|y_1 - y_2\right\|$$

$$+K_3\|z_1 - z_2\| + K_4\|w_1 - w_2\| \tag{4}$$

$$\|g(t,x_1,y_1,z_1,w_1) - g(t,x_2,y_2,z_2,w_2)\| \le Q_1\|x_1 - x_2\| + Q_2\|y_1 - y_2\|$$

$$+Q_3\|z_1 - z_2\| + Q_4\|w_1 - w_2\| \tag{5}$$

$$\left.\begin{aligned} \|Ax_1 - Ax_2\| &\le H_1\|x_1 - x_2\| \\ \|By_1 - By_2\| &\le H_2\|y_1 - y_2\| \end{aligned}\right\} \tag{6}$$

$for\ all\ t \in [0,T],\ \ x, x_1, x_2 \in D,\ \ y, y_1, y_2 \in D_1,\ \ z, z_1, z_2 \in D_2,\ \ w, w_1, w_2 \in D_3$

$where\ M_1, N_1, K_1, K_2, K_3, K_4, Q_1, Q_2, Q_3, Q_4, H_1\ and\ H_2$ are positive constants

We define the non-empty sets as follows :

$$\left.\begin{aligned} D_f &= D - M_1 T \\ D_{1f} &= D_1 - N_1 T \\ D_{2f} &= D_2 - H_1 M_1 T \\ D_{3f} &= D_3 - H_2 M_1 T \end{aligned}\right\} \tag{7}$$

We consider the matrix $\Lambda = \begin{pmatrix} \lambda_1 T & \lambda_2 T \\ \sigma_1 T & \sigma_2 T \end{pmatrix}$ (8)

Where $\lambda_1 = K_1 + K_3 H_1$ and $\lambda_2 = K_2 + K_4 H_2$

$$\sigma_1 = Q_1 + Q_3 H_1 \text{ and } \sigma_2 = Q_2 + Q_4 H_2 \tag{9}$$

Furthermore, we assume that the largest eigen value λ_max of the matix Λ does not exceed unity. that is :

$$\lambda_{\max}(\Lambda) = \frac{(\lambda_1 + \sigma_2)T + \sqrt{(\lambda_1 + \sigma_2)^2 T^2 - (\lambda_1 \sigma_{2-} \sigma_1 \lambda_2) T^2}}{2} \tag{10}$$

Define a sequence of functions $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$ on the domain :

$$(t, x_0, y_0) \in R^1 \times D_f \times D_{1f} \tag{11}$$

2.2. Main Results

In this section, include the existence, uniqueness and stability of the problem (1).

2.2.1. Existence Solution of (1)

In this section, we prove the existence theorem of differential equation (1) by using Picard approximation method :

Theorem 3.1 (Existence theorem). Let the vector functions $f(t, x, y, z, w)$ and $g(t, x, y, z, w)$ be defined on the domain (2), continuous in t,x,y,z,w and satisfies the inequalities (3) , (4) ,(5) and (6), has a solution $x = x(t, x_0)$ and $y = y(t, x_0)$ then there exist a sequence of functions.

$$x_{m+1}(\mathrm{t}, x_0, y_0) = x_0 + \int_0^t f(s, x_m(s), y_m(s), Ax_m(s), By_m(s))ds \tag{12}$$

and

$$y_{m+1}(\mathrm{t}, x_0, y_0) = y_0 + \int_0^t g(s, x_m(s), y_m(s), Ax_m(s), By_m(s))ds \tag{13}$$

with $x_0(0, x_0) = x_0$, $y_0(0, x_0) = y_0$and m =0,1,2,…

the limit function x (t, x_0, y_0) and y y (t, x_0, y_0) defined on the domain (2) and satisfy the following integral equations :

$$x\,(\mathrm{t}, x_0, y_0) = x_0 + \int_0^t f\big(s, x(s), y(s), Ax(s), By(s)\big)ds \tag{14}$$

$$y\,(\mathrm{t}, x_0, y_0) = y_0 + \int_0^t g\big(s, x(s), y(s), Ax(s), By(s)\big)ds \tag{15}$$

Proof.

From the sequence of functions (10) and (11), when m = 0 , we get

$$\left\|x_1(\mathrm{t}, x_0, y_0) - x_0\right\| = \left\| x_0 + \int_0^t f\big(s, x_0, y_0, Ax_0, By_0\big)ds - x_0 \right\|$$

$$\leq \int_0^t \| f(s, x_0, y_0, Ax_0, By_0)\| \, ds \leq \int_0^t M_1 \, ds \tag{16}$$

Therefore, we get

$$\left\|x_1(\mathrm{t}, x_0, y_0) - x_0\right\| \leq M_1 T \tag{17}$$

and similar , we have

$$\|y_1(\mathrm{t}, x_0, y_0) - y_0\| = \left\| y_0 + \int_0^t g(s, x_0, y_0, Ax_0, By_0)ds - y_0 \right\| \leq N_1 T \tag{18}$$

Therefore, $x_1(\mathrm{t}, x_0, y_0)$ and $y_1(\mathrm{t}, x_0, y_0)$∈D_1 .

Then by mathematical induction, we can obtain that

$$\|x_m(\mathrm{t}, x_0, y_0) - x_0 \| \,\delta\, M_1 T \tag{19}$$

and

$$\|y_m(\mathrm{t}, x_0, y_0) - y_0 \| \leq N_1 T \tag{20}$$

Also from (6) we get

$$\|Ax_m(\mathrm{t}, x_0, y_0) - Ax_0 \| \,\delta\, H_1 M_1 T \tag{21}$$

and

$$\|By_m(\mathrm{t}, x_0, y_0) - By_0 \| \,\delta\, H_2 N_1 T \tag{22}$$

That is $x_m(\mathrm{t}, x_0, y_0) \in D$, $y_m(\mathrm{t}, x_0, y_0) \in D_1$,, $Ax_m(\mathrm{t}, x_0, y_0) \in D_2$, $By_m(\mathrm{t}, x_0, y_0) \in D_3$, for all t ∈[0,T] and $x_0 \in D_f$, $y_0 \in D_{1f}$ $Ax_0(\mathrm{t}, x_0, y_0) \in D_{2f}$, $By_0(\mathrm{t}, x_0, y_0) \in D_{3f}$

Next, we shall to prove that the sequence of functions (10) and (11) converges uniformly on the domain (2) .

From (10) we put m =1 , we get :

$$\left\|x_2\left(\mathrm{t}, x_0, y_0\right) - x_1\left(\mathrm{t}, x_0, y_0\right) \right\| = \left\|x_0 + \int_0^t f\left(s, x_1(s), y_1(s), Ax_1(s), By_1(s)\right) ds\right.$$

$$\left. - x_0 - \int_0^t f(s, x_0, y_0, Ax_0, By_0) ds\right\|$$

$$\leq \int_0^t [\, K_1 \|x_1(\mathrm{s}, x_0, y_0) - x_0 \| + K_2 \|y_1(\mathrm{s}, x_0, y_0) - y_0 \|$$

$$+ K_3 \|Ax_1(\mathrm{t}, x_0, y_0) - \mathrm{A}\, x_0 \| + K_4 \|By_1(\mathrm{t}, x_0, y_0) - \mathrm{B}\, y_0 \| \,]\, ds$$

$$< T\,(K_1 + K_3 H_1) \|x_1(\mathrm{t}, x_0, y_0) - x_0 \| + T\,(K_2 + K_4 H_2) \|y_1(\mathrm{t}, x_0, y_0) - y_0 \|$$

$$< \lambda_1 T \,\|x_1(\mathrm{t}, x_0, y_0) - x_0 \| + \lambda_2 T \|y_1(\mathrm{t}, x_0, y_0) - y_0 \| \tag{23}$$

By mathematical induction and for all m≥1 ,the following inequality holds :

$$\left\|x_{m+1}\left(\mathrm{t}, x_0, y_0\right) - x_m\left(\mathrm{t}, x_0, y_0\right) \right\| \leq \lambda_1 T \left\|x_m\left(\mathrm{t}, x_0, y_0\right) - x_{m-1}\left(\mathrm{t}, x_0, y_0\right)\right\| + \lambda_2 T \|y_m(\mathrm{t}, x_0, y_0) - y_{m-1}(\mathrm{t}, x_0, y_0)\| \tag{24}$$

Also, from (11) we put m =1 , we get

$$\|y_2(\mathrm{t}, x_0, y_0) - y_1(\mathrm{t}, x_0, y_0) \| = \left\|y_0 + \int_0^t g\big(s, x_1(s), y_1(s), Ax_1(s), By_1(s)\big) ds\right.$$

$$\left. - y_0 - \int_0^t g(s, x_0, y_0, Ax_0, By_0) ds\right\|$$

$$\leq \int_0^t [\, Q_1 \|x_1(\mathrm{s}, x_0, y_0) - x_0 \| + Q_2 \|y_1(\mathrm{s}, x_0, y_0) - y_0 \|$$

$$+ Q_3 \|Ax_1(\mathrm{s}, x_0, y_0) - \mathrm{A}\, x_0 \| + Q_4 \|By_1(\mathrm{s}, x_0, y_0) - \mathrm{B}\, y_0 \| \,]\, ds$$

$$\leq T\,(Q_1 + Q_3 H_1) \|x_1(\mathrm{t}, x_0, y_0) - x_0 \| + T\,(Q_2 + Q_4 H_2) \|y_1(\mathrm{t}, x_0, y_0) - y_0 \|$$

$$\leq \sigma_1 T \,\|x_1(\mathrm{t}, x_0, y_0) - x_0 \| + \sigma_2 T \|y_1(\mathrm{t}, x_0, y_0) - y_0 \| \tag{25}$$

By mathematical induction and for all m≥1 ,the following inequality holds

$$\|y_{m+1}(\mathrm{t},x_0,y_0)-y_m(\mathrm{t},x_0,y_0)\|\ \delta\quad \sigma_1 T\,\|x_m(\mathrm{t},x_0,y_0)-x_{m-1}(\mathrm{t},x_0,y_0)\|+ \tag{26}$$

$$\sigma_2 T\|y_m(\mathrm{t},x_0,y_0)-y_{m-1}(\mathrm{t},x_0,y_0)\|$$

$$\begin{pmatrix}\|x_{m+1}(\mathrm{t},x_0,y_0)-\ x_m(\mathrm{t},x_0,y_0)\|\\ \|y_{m+1}(\mathrm{t},x_0,y_0)-y_m(\mathrm{t},x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}\lambda_1 T & \lambda_2 T\\ \sigma_1 T & \sigma_2 T\end{pmatrix}\begin{pmatrix}\|x_m(\mathrm{t},x_0,y_0)-\ x_{m-1}(\mathrm{t},x_0,y_0)\|\\ \|y_m(\mathrm{t},x_0,y_0)-y_{m-1}(\mathrm{t},x_0,y_0)\|\end{pmatrix} \tag{27}$$

That is :

$$V_{m+1}\left(\mathrm{t},x_0,y_0\right)\le\ \mathrm{E}\,(\tau)\,V_m\left(\mathrm{t},x_0,y_0\right) \tag{28}$$

$$V_{m+1}(\mathrm{t},x_0,y_0)=\begin{pmatrix}\|x_{m+1}(\mathrm{t},x_0,y_0)-\ x_m(\mathrm{t},x_0,y_0)\|\\ \|y_{m+1}(\mathrm{t},x_0,y_0)-y_m(\mathrm{t},x_0,y_0)\|\end{pmatrix} \tag{29}$$

$$\Lambda(t)=\begin{pmatrix}\llcorner_1 T & \llcorner_2 T\\ \sigma_1 T & \sigma_2 T\end{pmatrix} \tag{30}$$

and

$$V_m(\mathrm{t},x_0,y_0)=\begin{pmatrix}\|x_m(\mathrm{t},x_0,y_0)-\ x_{m-1}(\mathrm{t},x_0,y_0)\|\\ \|y_m(\mathrm{t},x_0,y_0)-y_{m-1}(\mathrm{t},x_0,y_0)\|\end{pmatrix} \tag{31}$$

which leads to the estimate

$$\textstyle\sum_{i=1}^{m}V_i\le\sum_{i=1}^{m}\Lambda^{i-1}\ V_1 \tag{32}$$

where $V_1=\begin{pmatrix}\lambda_1 T\\ \sigma_1 T\end{pmatrix}$, and $\Lambda=\underset{\mathrm{t}\in[0,T]}{max}\ \Lambda\ (\mathrm{t})$

Since $\lambda_{\max}(\Lambda)\ <\ 1$, Then the series (16) is uniformly convergent, i. e

$$\lim_{m\to\infty}\textstyle\sum_{i=1}^{m}\Lambda^{i-1}V_1=\sum_{i=1}^{\infty}\Lambda^{i-1}V_1=(E-\Lambda)^{-1}V_1 \tag{33}$$

The limiting relation (17) signifies a uniform convergence of the sequences of functions $x_m(\mathrm{t},x_0)$ and $y_m(\mathrm{t},x_0)$ on the domain (9) as m→∞

Putting

$$\left.\begin{aligned}\lim_{m\to\infty}x_m\,(t,x_0,y_0)=x(t,x_0,y_0)\\ \lim_{m\to\infty}y_m\,(t,x_0,y_0)=y(t,x_0,y_0)\end{aligned}\right] \tag{34}$$

Finally, we show that $x(\mathrm{t},x_0,y_0)\in D$ and $y(t,x_0,y_0)\in D_1$ for all $\mathrm{t}\in[0,\mathrm{T}]$, we take

$$\|x_m(\mathrm{t},x_0,y_0)-x\ (\mathrm{t},x_0,y_0)\|=\left\|\int_0^t f\left(s,x_m(s),y_m(s),Ax_m(s),By_m(s)\right)ds\ -\right.$$

$$\left.\int_0^t f\left(s,x(s),y(s),Ax(s),By(s)\right)ds\ \right\|$$

$$\le\ \ \lambda_1 T\,\|x_m(\mathrm{t},x_0,y_0)-\ x(t,x_0,y_0)\,\|+\ \ \lambda_2 T\|y_m(\mathrm{t},x_0,y_0)-\ y(t,x_0,y_0)\,\| \tag{35}$$

And

From (18), we assume that $\left\|x_m\left(\mathrm{t},x_0,y_0\right)-\ x(\mathrm{t},x_0,y_0)\ \right\|\ \le\in$ and

$$\|y_m(\mathrm{t},x_0,y_0)-\ y(\mathrm{t},x_0,y_0)\,\|\ \delta\in \tag{36}$$

Therefore,

$$\left\|x_m\left(\mathrm{t}, x_0, y_0\right) - x(\mathrm{t}, x_0, y_0)\right\| \delta \quad \lambda_1 T \in + \lambda_1 T \in$$

$$\left\|x_m\left(\mathrm{t}, x_0, y_0\right) - x(\mathrm{t}, x_0, y_0)\right\| \delta \in \mathrm{T}(\lambda_1 + \lambda_2)$$

$$\|x_m(\mathrm{t}, x_0, y_0) - x(\mathrm{t}, x_0, y_0)\| \,\delta \in_1 \tag{37}$$

for all $m \geq 0$, Putting $\in = \frac{\in_1}{(\lambda_1 + \lambda_2)T}$

Also in the same way we have

$$\|y_m(\mathrm{t}, x_0, y_0) - y(\mathrm{t}, x_0, y_0)\| \leq \sigma_1 T \|x_m(\mathrm{t}, x_0, y_0) - x(t, x_0, y_0)\| \tag{38}$$

$$+ \sigma_2 T \|y_m(\mathrm{t}, x_0, y_0) - y(t, x_0, y_0)\|$$

$$\left\|y_m\left(\mathrm{t}, x_0, y_0\right) - (\mathrm{t}, x_0, y_0)\right\| \delta \quad \sigma_1 T \in + \sigma_2 T \in$$

$$\left\|y_m\left(\mathrm{t}, x_0, y_0\right) - y(\mathrm{t}, x_0, y_0)\right\| \delta \in T(\sigma_1 + \sigma_2)$$

$$\|y_m(\mathrm{t}, x_0, y_0) - y(\mathrm{t}, x_0, y_0)\| \,\delta \in_2 \tag{39}$$

for all $m \geq 0$, Putting $\in = \frac{\in_2}{(\sigma_1 + \sigma_2)T}$

from (19) and (20) we get :

$$\begin{pmatrix} \|x_{m+1}(\mathrm{t}, x_0, y_0) - x_m(\mathrm{t}, x_0, y_0)\| \\ \|y_{m+1}(\mathrm{t}, x_0, y_0) - y_m(\mathrm{t}, x_0, y_0)\| \end{pmatrix} \leq \begin{pmatrix} \in_1 \\ \in_2 \end{pmatrix} \tag{40}$$

So that $x(\mathrm{t}, x_0, y_0) \in D$, $y(\mathrm{t}, x_0, y_0) \in D_1$ and hence $x(\mathrm{t}, x_0, y_0)$, $y(\mathrm{t}, x_0, y_0)$ are solutions of the problem (1).

2.2.2. Uniqueness Solution of (1)

Theorem 3.2 (Uniqueness Theorem)

Suppose that the hypotheses and all conditions and inequalities of the theorem 1, the solution $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$of the problem (1) are unique on the domain (2).

Proof . Let $\overline{x}\left(t, x_0, y_0\right)$and $\overline{y}\left(t, x_0, y_0\right)$ be another solutions of the problem (1) , then:

$$\overline{x}(t, x_0, y_0) = x_0 + \int_0^t f(s, \overline{x}(s), \overline{y}(s), A\overline{x}(s), B\overline{y}(s))ds$$

$$\overline{y}(t, x_0, y_0) = y_0 + \int_0^t g(s, \overline{x}(s), \overline{y}(s), A\overline{x}(s), B\overline{y}(s))ds \tag{41}$$

Thus, we have

$$\|x(\mathrm{t}, x_{0,}, y_0) - \overline{x}(t, x_0, y_0)\| = \left\|x_0 + \int_0^t f(s, x(s), y(s), Ax(s), By(s))ds - \right.$$

$$\left. x_0 - \int_0^t f(s, \overline{x}(s), \overline{y}(s), A\overline{x}(s), B\overline{y}(s))ds\right\|$$

$$\leq \lambda_1 T \|x(\mathrm{t}, x_0, y_0) - \overline{x}(t, x_0, y_0)\| + \lambda_2 T \|y(\mathrm{t}, x_0, y_0) - \overline{y}(t, x_0, y_0)\| \tag{42}$$

and the same way, we obtain that

$$\left\|y\left(\mathrm{t}, x_0, y_0\right) - \overline{y}\left(t, x_0, y_0\right)\right\| \leq \sigma_1 T \left\|x\left(\mathrm{t}, x_0, y_0\right) - \overline{x}\left(t, x_0, y_0\right)\right\| + \tag{43}$$

$$\sigma_2 T \|y(\mathrm{t}, x_0, y_0) - \overline{y}(t, x_0, y_0)\|$$

Rewrite the inequalities (21) and (22) in a vector form : -

$$\begin{pmatrix} \|x(\mathrm{t},x_0,y_0) - \bar{x}(t,x_0,y_0)\| \\ \|y(\mathrm{t},x_0,y_0) - \bar{y}(t,x_0,y_0)\| \end{pmatrix} \le \begin{pmatrix} \lambda_1 T & \lambda_2 T \\ \sigma_1 T & \sigma_2 T \end{pmatrix} \begin{pmatrix} \|x(\mathrm{t},x_0,y_0) - \bar{x}(t,x_0,y_0)\| \\ \|y(\mathrm{t},x_0,y_0) - \bar{y}(t,x_0,y_0)\| \end{pmatrix} \tag{44}$$

Therefore, we get

$$\begin{pmatrix} \|x(\mathrm{t},x_0,y_0) - \bar{x}(t,x_0,y_0)\| \\ \|y(\mathrm{t},x_0,y_0) - \bar{y}(t,x_0,y_0)\| \end{pmatrix} \le \Lambda^m \begin{pmatrix} \|x(\mathrm{t},x_0,y_0) - \bar{x}(t,x_0,y_0)\| \\ \|y(\mathrm{t},x_0,y_0) - \bar{y}(t,x_0,y_0)\| \end{pmatrix} \tag{45}$$

But from the condition $\lambda_{\max}(\Lambda) < 1$ as $m \longrightarrow \infty$, then

$$\begin{pmatrix} \|x(\mathrm{t},x_0,y_0) - \bar{x}(t,x_0,y_0)\| \\ \|y(\mathrm{t},x_0,y_0) - \bar{y}(t,x_0,y_0)\| \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{46}$$

And hence :

$$x(\mathrm{t},x_0,y_0) = \bar{x}(t,x_0,y_0)\,, \quad y\,(\mathrm{t},x_0,y_0) = \bar{y}(t,x_0,y_0) \tag{47}$$

Therefore, $x(\mathrm{t},x_0,y_0)$ and $y(t,x_0,y_0)$ are a unique solution of the problem (1).

2.2.3. Stability solution of (1)

In this section, we study the stability solution of the problem (1) by the following theorem:

Theorem 3.3 (Stability theorem)

Suppose that all the hypothesis and conditions of theorem 1 are satisfy, then the solutions of problem (1) is stable for all t > 0.

Proof. Let $z(\mathrm{t},x_0,y_0)$ and $\dot{z}(\mathrm{t},x_0,y_0)$be any different solutions of the problem (1),

$$z(\mathrm{t}) = z_0 + \int_0^t f(s,z(s),\dot{z}(\mathrm{s}),Az(s),B\dot{z}(\mathrm{s}))ds$$

$$\dot{z}(\mathrm{t}) = \dot{z}_0 + \int_0^t g(s,z(s),\dot{z}(\mathrm{s}),Az(s),B\dot{z}(\mathrm{s}))ds \tag{48}$$

Then, we have

$$\|x(t) - z(\mathrm{t})\| = \left\| x_0 + \int_0^t f(s,x(s),y(s),Ax(s),By(s))ds - z_0 - \int_0^t f(s,z(s),\dot{z}(\mathrm{s}),Az(s),B\dot{z}(\mathrm{s}))ds \right\| \tag{49}$$

$$\le \|x_0 - z_0\| + \lambda_1 T\,\|x(t) - z(\mathrm{t})\| + \lambda_2 T\|y(\mathrm{t}) - \dot{z}(\mathrm{t})\|$$

By the definition of stability for $\|x_0 - z_0\| \le \delta_1$ we get

$$\|x(t) - z(\mathrm{t})\| \le \delta_1 + \lambda_1 T\,\|x(t,x_0,y_0) - z(\mathrm{t},x_0,y_0)\| + \lambda_2 T\|y(\mathrm{t},x_0,y_0) - \dot{z}(\mathrm{t},x_0,y_0)\| \tag{51}$$

Similarly, we have

$$\|y(t) - \dot{z}(t)\| \le \|y_0 - \dot{z}_0\| + \sigma_1 T\,\|x(t) - z(\mathrm{t})\| + \sigma_2 T\|y(\mathrm{t}) - \dot{z}(\mathrm{t})\| \tag{52}$$

also by the definition of stability for $\|y - \dot{z}_0\| \le \delta_2$, we get

$$\|y(t) - \dot{z}(t)\| \le \delta_2 + \sigma_1 T\,\|x(t,x_0,y_0) - z(\mathrm{t},x_0,y_0)\| + \sigma_2 T\|y(\mathrm{t},x_0,y_0) - \dot{z}(\mathrm{t},x_0,y_0)\| \tag{53}$$

From (52) and (53) we have

$$\begin{pmatrix} \|x(\mathrm{t})-z(t)\| \\ \|y(t)-\dot{z}(t)\| \end{pmatrix} \leq \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{H}_1 T^2}{2} & \frac{\mathrm{H}_2 T^2}{2} \\ \mathrm{H}_1 T & \mathrm{H}_2 T \end{pmatrix} \begin{pmatrix} \|x(\mathrm{t})-z(t)\| \\ \|y(t)-\dot{z}(t)\| \end{pmatrix} \quad (54)$$

By the condition (8) and the definition of stability , we obtain that

$$\begin{pmatrix} \|x(\mathrm{t})-z(t)\| \\ \|y(t)-\dot{z}(t)\| \end{pmatrix} \leq \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad , \quad (\delta_1, \delta_2 > 0) \quad (55)$$

So, the solutions of (1) are stable for all $t \in [0,T]$

2.2.4. Existence and Uniqueness Solution of (1) by Banach fixed point theorem

In this section, we prove the existence uniqueness theorem of the problem (1) by using Banach fixed point theorem.

Theorem 3.4 (Banach Fixed Point Theorem).

Let $(x, \|.\|)$ be a Banach fixed point $H: x \to x$ be a contraction mapping Lipchitz continues with Lipchitz constant $L \in (0,1)$,then H has a unique fixed point.

Theorem 3.5 Assume that $S_x, S_y : R \, x \, X \, x \, X \, x \, X$, $X \to X$ is a Lipchitz continues functions satisfy the inequality (4) and (5) for all $t \in [0,T]$, also there exist constants $M_2$and N_2such that $M_2 = \sup|f(t,0,0,0,0)|$ and $N_2 = \sup|g(t,0,0,0,0)|$in additions assume that

$$T\,(\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\,\mathrm{H}_1 + \mathrm{k}_4\,\mathrm{H}_2 + \mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\,\mathrm{H}_1 + \mathrm{Q}_4\,\mathrm{H}_2) < 1 \quad (56)$$

Then there exist a unique solution of the problem (1) .

Proof : consider the operators S_x and $S_y$defined by the following

$$S_1 x(\mathrm{t}, x_0, y_0) = x_0 + \int_0^t f(x(s), y(s), z(s), w(s))ds \quad (57)$$

and

$$S_2 y(\mathrm{t}, x_0, y_0) = x_0 + \int_0^t g(x(s), y(s), z(s), w(s))ds \quad (58)$$

We show that $S\,B_r \; < B_r$ where $B_r = \{(x,y) \in X * X : \|(x,y)\| \leq r$ with

$$r \geq \frac{(\mathrm{M}_2 + \mathrm{N}_2)\,T}{1 - T\,(\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\,\mathrm{H}_1 + \mathrm{k}_4\,\mathrm{H}_2) - T\,(\mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\,\mathrm{H}_1 + \mathrm{Q}_4\,\mathrm{H}_2)} \quad (\chi 9)$$

for (x,y)∈x we have

$$\|S_1 x(\mathrm{t}, x_0, y_0)\| = \max_{t \in [0,T]} \left| \int_0^t f(s, x(s), y(s), z(s), w(s))ds - f(s,0,0,0,0) + f(s,0,0,0,0) \right| ds \quad (60)$$

$$\leq \int_0^t |\,\mathrm{k}_1 \|x\| + \mathrm{k}_2 \|y\| + \mathrm{k}_3\,\mathrm{H}_1 \|x\| + \mathrm{k}_4\,\mathrm{H}_2 \|y\| + \mathrm{M}_2|\,ds$$

$$\leq T(\mathrm{k}_1 + \mathrm{k}_3\,\mathrm{H}_1)\,\|x\| + T(\mathrm{k}_2 + \mathrm{k}_4\,\mathrm{H}_2)\,\|y\| + \mathrm{M}_2 T$$

hence

$$\|S_1 x(\mathrm{t}, x_0, y_0)\| \leq T\,(\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\,\mathrm{H}_1 + \mathrm{k}_4\,\mathrm{H}_2)\,r + \mathrm{M}_2 T \quad (61)$$

In the same way, we can obtain that

$$\|S_2 y\,(\mathrm{t}, x_0, y_0)\| \leq T\,(\mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\,\mathrm{H}_1 + \mathrm{Q}_4\,\mathrm{H}_2)\,r + \mathrm{N}_2 T \quad (62)$$

The equality $\|S\,(x\,,y\,)_{(t)}\| \leq r$

Now , for $(x_1 , y_1) , (x_2, y_2) \in X * Y$ and for any $t \in [0 , T]$, we get

$$\left\|S_1(x_2, y_2)_{(t)} - S_1(x_1, y_1)_{(t)}\right\| = \max_{t\in[0,T]} \left|\int_0^t f(s, x_2(s), y_2(s), Ax_2(s), By_2(s))\, ds - \int_0^t f(s, x_1(s), y_1(s), Ax_1(s), By_1(s))ds\right|$$

$$\leq T(\mathrm{k}_1\|x_2(t) - x_1(t)\| + \mathrm{k}_2\|y_2(t) - y_1(t)\| + \mathrm{k}_3\, \mathrm{H}_1\|x_2(t) - x_1(t)\| + \mathrm{k}_4\, \mathrm{H}_2\|y_2(t) - y_1(t)\|)$$

$$\leq T (\mathrm{k}_1 + \mathrm{k}_3\, \mathrm{H}_1)\, \|x_2(t) - x_1(t)\| + T\, (\mathrm{k}_2 + \mathrm{k}_4\, \mathrm{H}_2)\, \|y_2(t) - y_1(t)\| \quad (63)$$

Similarly, we obtain

$$\left\|S_2(x_2, y_2)_{(t)} - S_2(x_1, y_1)_{(t)}\right\| \leq T (\mathrm{Q}_1 + \mathrm{Q}_3\, \mathrm{H}_1)\, \|x_2(t) - x_1(t)\| + T\, (\mathrm{Q}_2 + \mathrm{Q}_4\, \mathrm{H}_2)\, \|y_2(t) - y_1(t)\| \quad (64)$$

Now the equations (25) and (26), we get

$$\left\|S_1(x_2, y_2)_{(t)} - S_1(x_1, y_1)_{(t)}\right\| \leq T\, (\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\, \mathrm{H}_1 + \mathrm{k}_4\, \mathrm{H}_2)\, (\|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|) \quad (65)$$

and

$$\left\|S_2(x_2, y_2)_{(t)} - S_2(x_1, y_1)_{(t)}\right\| \leq T\, (\mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\, \mathrm{H}_1 + \mathrm{Q}_4\, \mathrm{H}_2)\, (\|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|)$$

It follows (64) and (65) that

$$\|S(x_2, y_2) - S\, (x_1, y_1)\| \leq T\, (\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\, \mathrm{H}_1 + \mathrm{k}_4\, \mathrm{H}_2 + \mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\, \mathrm{H}_1 + \mathrm{Q}_4\, \mathrm{H}_2)\, (\|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|) \quad (66)$$

Since $T\, (\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3\, \mathrm{H}_1 + \mathrm{k}_4\, \mathrm{H}_2 + \mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_3\, \mathrm{H}_1 + \mathrm{Q}_4\, \mathrm{H}_2) < 1$

There fore , S is a contraction operator. So by Banach's fixed-point theorem the operator T has a unique fixed point, which is the unique solution of problem (1) this completes the proof

3. Results and Discussion

Nonlinear systems of differential equations are of significant importance for social informatics, as they allow for understanding and addressing complex social phenomena. These equations are invaluable for modelling intricate societal dynamics. Various social phenomena, including disease spread, opinion dynamics, and economic interactions, exhibit nonlinear behaviour. Nonlinear differential equations offer a reliable framework for modelling and analysing intricate systems, providing social informatics researchers with a more precise representation of real-world dynamics.

Additionally, nonlinear models are crucial for prediction and intervention strategies within the field of social informatics. Here, researchers often analyse vast datasets to gain a deeper understanding of human behaviour and societal trends. Nonlinear models are adept at capturing the fundamental dynamics of systems, which in turn allows for more precise forecasting and the development of effective intervention strategies for various social issues such as disease outbreaks, urban planning, and public policy design.

Moreover, these models help to understand the intricate feedback mechanisms that are ubiquitous in social systems. Social interactions are characterised by feedback loops and intricate relationships that may have consequential effects on social behaviour. Proficiency in non-linear modelling enables social informatics experts to comprehend and analyse these interactions, contributing to a deeper understanding of areas, such as social network dynamics, crowd behaviour, and decision-making processes.

Nonlinear dynamics, including chaos theory, provide an essential tool for identifying sudden and unexpected changes within social systems. This understanding is crucial in addressing complex problems in social informatics, such as the abrupt emergence of social movements or economic crises. By utilizing concepts from nonlinear dynamics, researchers in social informatics may determine critical transition and tipping points, thus facilitating more effective intervention and decision-making methods.

Nonlinear optimization techniques are additionally imperative in social informatics, permitting researchers to optimize resource allocation, decision-making processes, and the design of social policies. Social informatics students can employ these methods to tackle resource allocation challenges, enhance social welfare, and improve the operation of social networks.

Moreover, social informatics frequently necessitates interdisciplinary cooperation with specialists from fields such as epidemiology, economics, psychology, and town planning. Proficiency in non-linear systems provides social informatics experts with analytical and modelling tools required to cooperate efficiently and communicate intricate ideas across different domains.

Considering the ethical and social implications of technology and data use is a crucial aspect of social informatics. Nonlinear models are essential in evaluating the effects of technology and data-driven interventions on society, particularly in addressing the issues of bias, fairness, and privacy.

Nonlinear systems of differential equations are invaluable instruments in social informatics, enabling the modelling of intricate social phenomena, enhancing prediction accuracy, comprehending feedback mechanisms, identifying crucial transitions, optimising social systems, fostering interdisciplinary collaboration, and tackling ethical and social implications. Proficiency in non-linear systems enables social informatics experts to address significant societal problems and make a meaningful contribution to the advancement of society by providing data-driven insights and informed decision-making.

This field holds many opportunities for further exploration and expansion. Future studies can build upon the established foundations in several areas, including fostering interdisciplinary collaborations with experts from fields such as epidemiology, economics, psychology, and urban planning. Future studies can build upon the established foundations in several areas, including fostering interdisciplinary collaborations with experts from fields such as epidemiology, economics, psychology, and urban planning. Future studies can build upon the established foundations in several areas, including fostering interdisciplinary collaborations with experts from fields such as epidemiology, economics, psychology, and urban planning. This can enhance the application of nonlinear systems to tackle intricate societal challenges. Secondly, the adoption of data-driven methodologies and the integration of empirical data into nonlinear models can corroborate theoretical findings and augment the practical value of social informatics research. Additionally, the ethical and societal implications of technology and data usage are subject to further developments, demanding more thorough scrutiny. Nonlinear models can be instrumental in assessing these repercussions, guaranteeing that ethical factors are taken into account. Fourthly, complex networks, like social networks and transportation networks, provide opportunities for studying and optimizing complex systems using nonlinear dynamics. Additionally, dynamic policy design informed by nonlinear modelling could be a promising avenue for future research, providing decision-support systems that adapt to evolving social contexts. Nonlinear systems of differential equations will continue to be an essential instrument in our endeavour to navigate the diverse obstacles that exist in comprehending and improving the intricacies of our interconnected globe.

4. Conclusion

Nonlinear differential equations systems are of significant importance within social informatics. They present a robust framework to unravel the intricate dynamics behind a range of social phenomena. Nonlinear modeling provides an influential tool for social informatics researchers to scrutinize and comprehend complex systems. This not only enhances our understanding of real-world dynamics but also enables us to make informed decisions and formulate effective strategies to address urgent societal concerns. Areas of potential development in this discipline incorporate interdisciplinary partnerships, data-driven approaches, ethical deliberations, intricate network analysis, and adaptable policy formulation. These factors may enhance the utilisation of non-linear systems to tackle multifaceted societal issues and adjust to evolving social conditions.

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